

INTEGRATION OF PDES BY DIFFERENTIAL GEOMETRIC MEANS

NAGHMANA TEHSEEN AND GEOFF PRINCE

ABSTRACT. We use Vessiot theory and exterior calculus to solve partial differential equations (PDEs) of the type $u_{yy} = F(x, y, u, u_x, u_y, u_{xx}, u_{xy})$ and associated evolution equations. These equations are represented by the Vessiot distribution of vector fields. We develop and apply an algorithm to find the largest integrable sub-distributions and hence solutions of the PDEs. We then apply the integrating factor technique [19] to integrate this integrable Vessiot sub-distribution. The method is successfully applied to a large class of linear and non-linear PDEs.

1. INTRODUCTION

The geometric study of differential equations aims to describe the local and global structure of the solutions as integral submanifolds of an equation manifold. This study of PDEs leads to the concept of *Vessiot distributions*, which provides the convenient formal framework to investigate PDEs on an appropriate jet bundle.

In [19] the authors used exterior calculus and the Frobenius theory of foliations to show that integrating factors can be used to solve systems of higher order ordinary differential equations, and they considerably extended the class of group actions which could be thought of as symmetries of the system for the purpose of reduction to the quadratures. Moreover their methods avoided cumbersome changes of coordinates, allowing solutions and first integrals to be represented in the original, physical coordinates of the problem. Later this technique is extended to the first order PDEs in [2, 3, 4, 5]. In the present paper we will extend the integrating factor technique to second order PDEs by using their results to integrate the (integrable sub-distributions of the) vector field description of the PDEs.

The formulation of a (finite) system of differential equations as a distribution of vector fields on some appropriate jet space is due to Vessiot [22, 23, 24] and called a Vessiot distribution. The maximal integrable sub-distributions of the Vessiot distribution represent every local solution of the PDE system. Computing the Vessiot distribution is routine but finding the maximal integrable sub-distributions is not. In fact there is no known algorithm for computing them in the generic case.

Solving the PDE system is equivalent to integrating a Pfaffian system or its dual vector field system. The Vessiot formulation of PDEs is dual to the more popular *exterior differential system (EDS)* formulation due to Cartan [6] and elaborated in [13, 15]. In [9] the author pointed out that the former is in many ways computationally simpler because vector fields are simpler to implement than the full exterior calculus of p -forms. Recent developments in the Vessiot theory can be found in [10, 20, 21].

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The purpose of this paper is to provide a systematic approach to solving second order linear and non-linear PDEs in the presence of symmetries. We use a general class of symmetries known as *solvable structures* which are not necessarily of point type. To generate the solvable structures for each integrable sub-distributions we use the symmetry determination software package DIMSYM [17] operating as a REDUCE [12] overlay. We also use the exterior calculus package EXCALC [16].

The paper is organised as follows. In Section 2 we introduce the key concepts of the Vessiot distribution. In Section 3 we give an explanation of how a reduction of order is possible using symmetries and present the main result of [19] which we will use to integrate the Frobenius integrable distribution of vector fields. In Section 4 we describe, in detail, a three steps algorithm to find the largest integrable sub-distributions which satisfy the appropriate independence/projectability condition. To integrate such distributions we use the results of [19]. We present some examples to explain the algorithm. In Section 5 we produce an EDS method to solve the differential conditions in step 2 of the algorithm. We also demonstrate the construction of a group invariant solution of our second order PDEs. We will explain the methods with some concrete examples.

2. BASIC CONCEPTS AND NOTATIONS

Consider a system of partial differential equations [20] of m independent variables x^i and n dependent variables u^j ,

$$(1) \quad F^a(x^i, u^j, u_{i_1}^j, u_{i_1 i_2}^j \dots u_{i_1 \dots i_k}^j) = 0.$$

Denote by X the space of independent variables and by U the space of dependent variables. The subscripts $1 \leq i_1 \leq \dots \leq i_k \leq n$ are used to specify the partial derivatives of u^j , where k is the maximum order of the system. The reader may consult [20] for more details.

Consider the trivial bundle $\pi: X \times U \rightarrow X$ and let f and g be two smooth sections of π . We say that f and g are equivalent to order k at x if and only if

$$\frac{\partial^{q_1 + \dots + q_m} f}{(\partial x^1)^{q_1} \dots (\partial x^m)^{q_m}}(x) = \frac{\partial^{q_1 + \dots + q_m} g}{(\partial x^1)^{q_1} \dots (\partial x^m)^{q_m}}(x)$$

for all m -tuples $(q_1 \dots q_m)$ with $q_1 + \dots + q_m \leq k$. The equivalence class of a smooth section f at a point x is the k -jet of f at x and is denoted by $j_x^k f$. The set of all k -jets of smooth sections of π constitutes the bundle of k -jets of maps $X \rightarrow U$ and denoted by $J^k(X, U)$. The zeroth order jet bundle, $J^0(X, U)$, is identified with $X \times U$. The k -graph of smooth section f of π is the map $j^k f: X \rightarrow J^k(X, U)$ defined by $x \rightarrow j_x^k f$. Thus, the image of k -graph of a section is a m -dimensional immersed submanifold of $J^k(X, U)$.

The geometry of jet bundles is to a large extent determined by their *contact structure*. Each jet bundle $J^k(X, U)$ comes equipped with a module of differential 1-forms spanned by

$$\theta_I^j := du_I^j - \sum_{i=1}^m u_{I,i}^j dx^i,$$

where I is a multi-index of order less than or equal to $k-1$ and is denoted by $\Omega^k(X, U)$. The contact system $\Omega^k(X, U)$ on J^k is also an example of an exterior differential system generating an *ideal* in the ring of differential forms on a differentiable manifold. This particular differential system generated by 1-forms is also called a *Pfaffian system*.

Another way to view this is that there may be a submanifold $N \subset M$ where all the forms in the contact co-distribution are zero when pulled back to N . That is, if $i : X \rightarrow J^k(X, U)$ is any immersion whose pull-back annihilates the contact system and satisfies the *transverse condition* or *independence condition* $i^*(dx^1 \wedge \dots \wedge dx^n) \neq 0$, then $i(X)$ is the image of some k -jet. The image of the k -jet of any function $f : X \rightarrow U$ is an *integral submanifold* of the k th order contact system.

Now we can see how these constructions are related to differential equations. The study of second and higher order differential equations on manifolds needs to consider jet bundles on some underlying space. In view of the definition of a jet bundle, the k -jet bundle $J^k(X, U)$ is the appropriate framework for dealing with the systems of PDEs of a higher order of m dependent variables and n independent variables. We will be interested in the second order partial differential equations which define embedded submanifolds of $J^k(X, U)$.

A solution of (1) is a smooth function $f : X \rightarrow U$ whose k -graph J_f^k defines a m -dimensional submanifold S of a smooth embedded submanifold of $J^k(X, U)$.

3. IDEALS, SYMMETRIES AND REDUCTION OF ORDER

We will briefly explain how the integrating factor technique in [19] allows a reduction of order via symmetries for higher order ordinary differential equations. We begin with some basics, for more details see [4, 5, 7, 15, 19].

Let M be some smooth manifold of dimension n . For simplicity, we suppose that all our objects are smooth on M . The algebra of differential forms on M , $\Lambda(M)$, is a graded algebra. An *ideal* I in $\Lambda(M)$ is an additive subgroup of $\Lambda(M)$ that is closed under the wedge product ($\beta^i \in I$ implies $\beta^i \wedge \alpha \in I, \forall \alpha \in \Lambda(M)$). An ideal I is a *differential ideal* if the exterior derivative of every member of I is also in I . A vector field V is said to be a *symmetry* of an ideal if $\mathcal{L}_V I \subset I$.

The *kernel* of a differential form Ω is the submodule of vector fields annihilating Ω , $\ker \Omega := \{Y \in \mathfrak{X}(M) : Y \lrcorner \Omega = 0\}$. A differential p -form Ω on M is *simple* or *decomposable* if it is the wedge product of p 1-forms, so Ω is locally simple if $\ker \Omega$ is everywhere $n - p$ dimensional, that is, of maximal dimension. A *constraint* 1-form θ for differential form Ω is a 1-form satisfying $\theta \wedge \Omega = 0$, which implies $Y \lrcorner \theta = 0, \forall Y \in \ker \Omega$. An m -dimensional distribution D on M is an assignment of an m -dimensional subspace D_x of $T_x M$ to each point x of M . The corresponding co-distribution D^\perp consists of the annihilators $D_x^\perp \subset T_x^* M$. As a submodule of $\Lambda^1(M)$ the elements of D^\perp are called constraint forms for D . A *characterising* form for an m -dimensional distribution D is a form Ω on M of degree $(n - m)$ which is the exterior product of $(n - m)$ constraint forms.

A distribution D has (maximal) integral submanifolds if and only if

$$(2) \quad d\theta \wedge \Omega = 0, \quad \forall \theta \in D^\perp \quad (\Omega \text{ is any characterising form for } D)$$

or

$$[Y, Z] \in D, \quad \forall Y, Z \in D.$$

In this case D is said to be *involutive* or *Frobenius integrable* and any characterising form Ω is said to be Frobenius integrable. Ω is Frobenius integrable if its kernel is Frobenius integrable and of maximal dimension everywhere. A function f on M is called an *integrating factor* for

the Frobenius integrable 1-form θ if $d(f\theta) = 0$. A vector field $V \in \mathfrak{X}(M)$ is a symmetry of a vector field distribution $D \subset \mathfrak{X}(M)$ if $\mathcal{L}_V D \subset D$ and V is a symmetry of differential form Ω if $\mathcal{L}_V \Omega = \lambda \Omega$ for some smooth function λ on M . If V is in D it is said to be *trivial*. For any vector field distribution D , we say that a collection of k linearly independent vector fields $X_1, \dots, X_k \in \mathfrak{X}(M)$ forms a *solvable structure* for D if

$$\begin{aligned} \mathcal{L}_{X_k} D &\subset D, \\ \mathcal{L}_{X_{k-1}} (\text{Sp}\{X_k\} \oplus D) &\subset \text{Sp}\{X_k\} \oplus D, \\ &\vdots \\ \mathcal{L}_{X_1} (\text{Sp}\{X_2, \dots, X_k\} \oplus D) &\subset \text{Sp}\{X_2, \dots, X_k\} \oplus D. \end{aligned}$$

DIMSVM can be used to locate solvable structures for distributions. The following results will be useful.

Theorem 1. [19] *Let θ be Frobenius integrable 1-form which is nowhere zero on some open subset U of a manifold M . Then μ is an integrating factor for θ if and only if $\mu = (X \lrcorner \theta)^{-1}$ for some symmetry vector field X on U with $X \lrcorner \theta \neq 0$ on U .*

Proposition 2. Let $\theta^1, \theta^2, \dots, \theta^k \in \Lambda^1(M)$ then $\text{Sp}\{\theta^1, \theta^2, \dots, \theta^k\}$ is Frobenius integrable,

$$d\theta^a \wedge \theta^1 \wedge \dots \wedge \theta^k = 0, \quad a = 1, \dots, k,$$

if and only if $d(\theta^1 \wedge \dots \wedge \theta^k) = \lambda \wedge \theta^1 \wedge \dots \wedge \theta^k$, for some 1-form λ .

Proof. If $\text{Sp}\{\theta^1, \theta^2, \dots, \theta^k\}$ is Frobenius integrable, then $d\theta^a \wedge \theta^1 \wedge \dots \wedge \theta^k = 0$, and from the definition of exterior derivative $d(\theta^1 \wedge \dots \wedge \theta^k) = d\theta^1 \wedge \theta^2 \wedge \dots \wedge \theta^k + \dots + (-1)^k \theta^1 \wedge \dots \wedge d\theta^k$. It is obvious from the Frobenius conditions that $(\theta^1 \wedge \dots \wedge \theta^k)$ divides $d(\theta^1 \wedge \dots \wedge \theta^k)$. Hence, $d(\theta^1 \wedge \dots \wedge \theta^k) = \lambda \wedge \theta^1 \wedge \dots \wedge \theta^k$.

Conversely, assume that $d(\theta^1 \wedge \dots \wedge \theta^k) = \lambda \wedge \theta^1 \wedge \dots \wedge \theta^k$ then, by using the definition of exterior derivative,

$$d\theta^1 \wedge \theta^2 \wedge \dots \wedge \theta^k + \dots + (-1)^k \theta^1 \wedge \dots \wedge d\theta^k = \lambda \wedge \theta^1 \wedge \dots \wedge \theta^k.$$

By taking the wedge product with θ^a , we have

$$d\theta^a \wedge \theta^1 \wedge \dots \wedge \theta^k = 0.$$

So, $\text{Sp}\{\theta^1, \dots, \theta^k\}$ is Frobenius integrable. □

We now use Proposition 2 to give a generalisation of Theorem 1 (compare with [5]).

Theorem 3. *Let Ω be a k -form on a manifold M , and $\text{Sp}(\{X_1 \dots X_k\})$ be a k -dimensional distribution on open $U \subseteq M$ satisfying $X_i \lrcorner \Omega \neq 0$ everywhere on U . Further suppose that $\text{Sp}(\{X_1 \dots X_k\} \cup \ker \Omega)$ is Frobenius integrable for some $j < k$ and that X_i is a symmetry of $\text{Sp}(\{X_{i+1} \dots X_k\} \cup \ker \Omega)$ for $i = 1 \dots j$.*

Put $\alpha^i := X_1 \lrcorner \dots \lrcorner \bar{X}_i \lrcorner \dots \lrcorner X_k \lrcorner \Omega$, where \bar{X}_i indicates that this argument is missing, and put $\omega^i := \frac{\alpha^i}{X_i \lrcorner \alpha^i}$ for $i = 1 \dots k$, so that $\{\omega^1 \dots \omega^k\}$ is dual to $\{X_1 \dots X_k\}$. Then

- (i) $\omega^1 \wedge \omega^2 \wedge \dots \wedge \omega^k = \frac{\Omega}{\Omega(X_1, X_2 \dots X_k)}$
- (ii) $\omega^1 \wedge \omega^2 \wedge \dots \wedge \omega^k$ is closed.

Proof. To prove the first part, observe that

$$\omega^1 \wedge \omega^2 = - \frac{(X_1 \lrcorner \Omega) \wedge (X_2 \lrcorner \Omega)}{(\Omega(X_2, X_1))^2}.$$

We know that $(X_1 \lrcorner \Omega) \wedge (X_2 \lrcorner \Omega) = X_1 \lrcorner (\Omega \wedge (X_2 \lrcorner \Omega)) - (X_1 \lrcorner X_2 \lrcorner \Omega) \Omega$. As the 2-form Ω is characterising form and $X_2 \lrcorner \Omega$ is a linear combination of θ and ϕ , so that $\Omega \wedge (X_2 \lrcorner \Omega) = 0$ implies that $(X_1 \lrcorner \Omega) \wedge (X_2 \lrcorner \Omega) = -(X_1 \lrcorner X_2 \lrcorner \Omega) \Omega$. Thus $\omega^1 \wedge \omega^2 = \frac{\Omega}{\Omega(X_2, X_1)}$. Clearly the process is inductive, and then the proof is complete.

To prove the second part, let

$$\begin{aligned} (\Omega(X_2, X_1))^2 d\left(\frac{\Omega}{\Omega(X_2, X_1)}\right) &= (\Omega(X_2, X_1))^2 \left(-\frac{d\Omega(X_2, X_1) \wedge \Omega}{(\Omega(X_2, X_1))^2} + \frac{d\Omega}{\Omega(X_2, X_1)}\right) \\ &= -d\Omega(X_2, X_1) \wedge \Omega + \Omega(X_2, X_1) d\Omega \\ &= d(X_2 \lrcorner X_1 \lrcorner \Omega) \wedge \Omega + \Omega(X_2, X_1) d\Omega \end{aligned}$$

Using $\mathcal{L}_{X_2} \alpha = X_2 \lrcorner d\alpha + d(X_2 \lrcorner \alpha)$, we have

$$\begin{aligned} (\Omega(X_2, X_1))^2 d\left(\frac{\Omega}{\Omega(X_2, X_1)}\right) &= [\mathcal{L}_{X_2}(X_1 \lrcorner \Omega) - X_2 \lrcorner d(X_1 \lrcorner \Omega)] \wedge \Omega + \Omega(X_2, X_1) d\Omega \\ &= -(X_2 \lrcorner (\mathcal{L}_{X_1} \Omega) \wedge \Omega + (X_2 \lrcorner X_1 \lrcorner d\Omega) \wedge \Omega + \Omega(X_2, X_1) d\Omega \\ &= -X_1 \lrcorner (X_2 \lrcorner d\Omega) \wedge \Omega + (X_2 \lrcorner d\Omega) \wedge (X_1 \lrcorner \Omega) + \Omega(X_2, X_1) d\Omega \end{aligned}$$

Now use Proposition 2,

$$\begin{aligned} (\Omega(X_2, X_1))^2 d\left(\frac{\Omega}{\Omega(X_2, X_1)}\right) &= -X_1 \lrcorner [X_2 \lrcorner (K \wedge (X_1 \lrcorner \Omega) \wedge (X_2 \lrcorner \Omega))] \wedge \Omega + \Omega(X_2, X_1) d\Omega \\ &\quad + [X_2 \lrcorner (K \wedge (X_1 \lrcorner \Omega) \wedge (X_2 \lrcorner \Omega))] \wedge (X_1 \lrcorner \Omega) \\ &= -\Omega(X_2, X_1) d\Omega + \Omega(X_2, X_1) d\Omega \\ &= 0. \end{aligned}$$

Hence, $d(\omega^1 \wedge \omega^2) = 0 = d\left(\frac{\Omega}{\Omega(X_2, X_1)}\right)$. The proof then follows by induction. \square

The paper by Sherring and Prince [19] extends Lie's approach to integrating a Frobenius integrable distribution by using solvable structures. The important results of [19] are reproduced below. We illustrate these results with the straightforward case of a single second order ordinary differential equation.

On some open subset of the first order jet bundle $J^1(\mathbb{R}, \mathbb{R})$ with coordinates $x, y, p = \frac{dy}{dx}$, the second order differential equation (vector) field of the differential equation $\frac{d^2y}{dx^2} = f(x, y, p)$ is

$$(3) \quad \Gamma := \frac{\partial}{\partial x} + p \frac{\partial}{\partial y} + f \frac{\partial}{\partial p}.$$

This has dual distribution spanned by the contact form $\theta = dy - p dx$ and the force form $\phi = dp - dx$. The 2-form $\Omega = \theta \wedge \phi$ on U is then a characterising form for (3).

Proposition 4. [19] If X is a vector field whose span with Γ is two dimensional on U then the 1-form $X \lrcorner \Omega$ is Frobenius integrable if and only if X and Γ are involution.

Corollary 5. [19] *If X is a non-trivial symmetry of (3) then $X \lrcorner \Omega$ is Frobenius integrable.*

Corollary 6. [19] *If X_1 and X_2 are symmetries of (3) which commute and whose span with Γ is three dimensional then the two forms*

$$\frac{X_1 \lrcorner \Omega}{X_2 \lrcorner X_1 \lrcorner \Omega} \quad \text{and} \quad \frac{X_2 \lrcorner \Omega}{X_1 \lrcorner X_2 \lrcorner \Omega}$$

are closed and locally provide a complete set of two functionally independent first integrals.

The two first integrals of this corollary, together with x , provide a coordinate chart which straightens out the integral curves of Γ .

The next corollary to Proposition 4 shows how a solvable structure of two symmetries produces one closed form ω^1 and a second form ω^2 which is closed modulo ω^1 , that is $d\omega^2 \equiv 0 \pmod{\omega^1}$ or locally $\omega^1 = d\gamma^1$ and $\omega^2 = d\gamma^2 + \gamma^0 d\gamma^1$. This result in [19] is proved by using the Frobenius theory of foliations but we will prove it in a very simple way by using exterior calculus. We need Theorem 3 and the following lemma.

Lemma 1. Let $\alpha, \beta, df \in \Lambda^1(M)$ be non-zero one-forms with $\beta \notin \text{Sp}\{df\}$. Then $d\alpha = df \wedge \beta$ if and only if $\beta \equiv dg \pmod{df}$ for some $g \in \Lambda^0(M)$. Equivalently, $d\beta = \lambda \wedge df \iff \beta \equiv dg \pmod{df}$.

Proof. Suppose that $d\alpha = df \wedge \beta$. Then it is easy to see that $D^\perp := \text{Sp}\{df, \beta\}$ is Frobenius integrable and so locally there exists dh such that $D^\perp := \text{Sp}\{df, dh\}$. Hence, $\beta = m df + k dh$, $m, k \in \Lambda^0$ and $d\alpha = df \wedge \beta = df \wedge (m df + k dh) = df \wedge k dh$. Now $0 = d^2\alpha = d(df \wedge k dh) = -df \wedge dk \wedge dh$, so that $dk \in \text{Sp}\{df, dh\}$. This implies k is a composite of f and h alone. Let g be the partial integral of k with respect to h , then $k dh = \frac{\partial g}{\partial h} dh$ and $d\alpha = df \wedge k dh = df \wedge (\frac{\partial g}{\partial h} dh) = df \wedge (\frac{\partial g}{\partial h} dh + \frac{\partial g}{\partial f} df) = df \wedge dg$. Hence, $df \wedge \beta = df \wedge dg$ which implies that $df \wedge (\beta - dg) = 0$ and $\beta \equiv dg \pmod{df}$ or $\beta = dg + l df$. The converse is trivial. \square

Corollary 7. *If X_1 is a symmetry of (3), X_2 is a symmetry of $\text{Sp}\{X_1, \Gamma\}$ and X_1, X_2 and Γ are linearly independent everywhere then*

$$\omega^1 := \frac{X_1 \lrcorner \Omega}{X_2 \lrcorner X_1 \lrcorner \Omega}$$

is closed, while

$$\omega^2 := \frac{X_2 \lrcorner \Omega}{X_1 \lrcorner X_2 \lrcorner \Omega}$$

is closed modulo ω^1 . These two forms locally provide a complete set of two functionally independent first integrals, $\omega^1 = d\gamma^1$ and $\omega^2 = d\gamma^2 - X_2(\gamma^2)d\gamma^1$. By putting $\hat{X}_2 = X_2 - X_2(\gamma^2)X_1$ we then have the two commuting symmetries X_1 and \hat{X}_2 which provide two first integrals γ^1 and γ^2 via Corollary 6.

Proof. The kernel of $X_1 \lrcorner \Omega$ is spanned by X_1 and Γ and $X_1 \lrcorner \Omega$ is Frobenius integrable, since $\text{Sp}(\{X_1, \Gamma\})$ is closed under lie bracket. So, ω^1 is closed by Theorem 1. To see that ω^2 is closed modulo ω^1 , we consider $\omega^1 \wedge \omega^2 = \frac{\Omega}{\Omega(X_2, X_1)}$ and by using the fact $d(\omega^1 \wedge \omega^2) = 0$ (see proof in Theorem 3), this implies that $d(\omega^1 \wedge \omega^2) = 0 = d(d\gamma^1 \wedge \omega^2) = d\omega^2 \wedge d\gamma^1$.

If $d\omega^2 \wedge d\gamma^1 = 0$, from Theorem 1 we may write $d\omega^2 = \mu \wedge d\gamma^1$ for some $\mu \in \Lambda^0$, so we have from Lemma 1 that $\omega^2 = d\gamma^2 + l d\gamma^1$. To find the value of l , contract this equation with X_1 and X_2 , we get $X_1(\gamma^2) = 1$ and $l = -X_2(\gamma^2)$. Hence, $\omega^2 = d\gamma^2 - X_2(\gamma^2)d\gamma^1$. \square

Corollary 7 can be generalised to the following theorem.

Theorem 8. [19] *Let Ω be a k -form on a manifold M , and let $Sp(\{X_1, \dots, X_k\})$ be a k -dimensional distribution on an open $U \subseteq M$ satisfying $X_i \lrcorner \Omega \neq 0$ everywhere on U . Further suppose that $Sp(\{X_{j+1}, \dots, X_k\} \cup \ker \Omega)$ is integrable for some $j < k$ and that X_i is a symmetry of $Sp(\{X_{i+1}, \dots, X_k\} \cup \ker \Omega)$ for $i = 1, \dots, j$.*

Put $\sigma^i := X_1 \lrcorner \dots \lrcorner \bar{X}_i \lrcorner \dots \lrcorner X_k \lrcorner \Omega$, where \bar{X}_i indicates that this argument is missing and $\omega^i := \frac{\sigma^i}{X_i \lrcorner \sigma^i}$ for $i = 1, \dots, k$ so that $\{\omega^1, \dots, \omega^k\}$ is dual to $\{X_1, \dots, X_k\}$. Then $d\omega^1 = 0$; $d\omega^2 = 0 \pmod{\omega^1}$; $d\omega^3 = 0 \pmod{\omega^1, \omega^2}$; \dots ; $d\omega^j = 0 \pmod{\omega^1, \dots, \omega^{j-1}}$, so that locally

$$\begin{aligned} \omega^1 &= d\gamma^1, \\ \omega^2 &= d\gamma^2 - X_1(\gamma^2)d\gamma^1, \\ \omega^3 &= d\gamma^3 - X_2(\gamma^3)d\gamma^2 - (X_1(\gamma^3) - X_2(\gamma^3)X_1(\gamma^2))d\gamma^1, \\ &\vdots \\ \omega^j &= d\gamma^j \pmod{d\gamma^1, \dots, d\gamma^{j-1}}, \end{aligned}$$

*for some $\gamma^1, \dots, \gamma^j \in \Lambda^0 T^*U$. Also the system $\{\omega^{j+1}, \dots, \omega^k\}$ is integrable modulo $d\gamma^1, \dots, d\gamma^j$ and locally $\Omega = \gamma^0 d\gamma^1 \wedge d\gamma^2 \wedge \dots \wedge d\gamma^j \wedge \omega^{j+1} \wedge \dots \wedge \omega^k$ for some $\gamma^0 \in \Lambda^0(T^*U)$. Each γ^i is uniquely defined up to the addition of arbitrary function of $\gamma^1, \dots, \gamma^{i-1}$.*

Sherring [18] gives an algorithm for the integration of these forms which we reproduce in Theorem 9 below. We repeatedly use this process in our treatment of PDEs in subsequent sections. The integration of an exact 1-form is described in the following lemma.

Lemma 2. Given an exact 1-form $\omega := \omega_i dx^i$ on \mathbb{R}^n , if $\sigma^i := \int (\omega_i - \sum_{j=1}^{i-1} \sigma_{x_i}^j) dx^i$ and $\gamma := \sum_{i=1}^n \sigma^i$ then $\omega = d\gamma$.

Proof. Let $\omega = d\gamma$ for some γ as ω is exact and so $\omega_i dx^i = \frac{\partial \gamma}{\partial x^i} dx^i$. For $i = 1$, $\gamma = \int \omega_1 dx^1 + \phi^1(x^2, \dots, x^n) = \sigma^1 + \phi^1(x^2, \dots, x^n)$ and then, $\gamma_{x^2} = \sigma_{x^2}^1 + \phi_{x^2}^1 = \omega_2$ this implies that, $\phi^1 = \int (\omega_2 - \sigma_{x^2}^1) dx^2 + \phi^2(x^3, \dots, x^n)$ and $\gamma = \sigma^1 + \sigma^2 + \phi^2(x^3, \dots, x^n)$. Now assume this holds for $i = k$, $\gamma = \sum_{i=1}^k \sigma^i + \phi^k(x^{k+1}, \dots, x^n)$. Then the result holds for $i = k + 1$, $\gamma = \sum_{i=1}^{k+1} \sigma^i + \phi^k(x^{k+2}, \dots, x^n)$. The claim then follows by induction and the n^{th} claim proves the theorem as ϕ^n is just the constant of integration. \square

Theorem 9. Let $\omega \in \Lambda(\mathbb{R}^n)$ which is exact modulo $d\gamma^1, \dots, d\gamma^j$. Without loss of generality choose coordinates $x^l := \gamma^l$, for $l = 1, \dots, j$ and $y^k := x^k, k = j + 1, \dots, n$. Then $\omega := \omega_i dx^i$, put $\sigma^i := \int (\omega_i - \sum_{k=j+1}^{i-1} \sigma_{x_i}^k) dx^i$ for $i = j + 1, \dots, n$ and $\gamma := \sum_{i=j+1}^n \sigma^i$. Then $\omega \equiv d\gamma \pmod{d\gamma^1 \dots d\gamma^j}$.

Proof. Given that ω is a non zero 1-form which is exact modulo $d\gamma^1, \dots, d\gamma^j$. Clearly, by choosing coordinates $x^l = \gamma^l$, $l = 1, \dots, j$; we then have ω is exact. Now $\omega_i = \frac{\partial \gamma}{\partial x^i}$, for $i = j + 1$,

$$\gamma = \int \omega_{j+1} dx^{j+1} + \phi^{j+1}(x^{j+2}, \dots, x^n) = \sigma^{j+1} + \phi^{j+1}(x^{j+2}, \dots, x^n)$$

and then, $\gamma_{x^{j+2}} = \sigma_{x^{j+2}}^{j+1} + \phi_{x^{j+2}}^{j+1} = \omega_{j+2}$ this implies that,

$$\phi^{j+1} = \int (\omega_{j+2} - \sigma_{x^{j+2}}^{j+1}) dx^{j+2} + \phi^{j+2}(x^{j+3}, \dots, x^n)$$

and

$$\gamma = \sigma^{j+1} + \sigma^{j+2} + \phi^{j+2}(x^{j+3}, \dots, x^n).$$

Clearly, for $i = j + 3$, $\gamma = \sum_{k=j+1}^{j+3} \sigma^k + \phi^{j+4}(x^{j+4}, \dots, x^n)$ and if we continue this process, $\gamma = \sum_{i=j+1}^n \sigma^i$ as ϕ^n is just the constant of integration. \square

For ordinary differential equations the vector field description is one-dimensional, and hence integrable by dimension. For PDEs the situation is different. And so having explained the integration of a Frobenius integrable distribution via solvable structures we now turn to the issue of locating integrable sub-distributions of non-integrable distributions.

We begin with a non-integrable distribution D of constant dimension p on an n -dimensional manifold M . Let D^\perp be the $n - p$ dimensional co-distribution and let D^* be a complementary distribution with $\mathfrak{X}(M) = D \oplus D^*$. We aim to find the largest integrable sub-distributions of D , of dimension $p - M$ say, which satisfy the appropriate independence condition.

This is done in stages by using the following algorithm. We will try to reduce the dimension of D^\perp and correspondingly add more 1-forms to the distribution D . Then the integrability condition (2) for the augmented distribution is divided into two parts, one is algebraic and the other is differential

$$(4) \quad d\sigma^a \wedge \Omega_\sigma \wedge \Omega_\rho = 0, \quad a = 1, \dots, n - p$$

$$(5) \quad d\rho^\alpha \wedge \Omega_\sigma \wedge \Omega_\rho = 0, \quad \alpha = 1, \dots, M \leq p.$$

Here $\Omega_\sigma := \sigma^1 \wedge \dots \wedge \sigma^{n-p}$ is a characterising form for D and ρ^α are 1-forms in $D^{*\perp}$ with $\Omega_\rho := \rho^1 \wedge \dots \wedge \rho^M$ and $\Omega_\sigma \wedge \Omega_\rho \neq 0$.

Firstly, we will add a single 1-form $\rho \in D^{*\perp}$ to D^\perp and generate the algebraic conditions (4) for $\text{Sp}\{\sigma^1, \dots, \sigma^p, \rho\}$ to be Frobenius integrable. If the algebraic conditions can be satisfied then there may be integrable Vessiot sub-distributions of dimension $p - 1$. If the algebraic conditions fails then we will add two 1-forms from $D^{*\perp}$. The process continues until we have an enlarged co-distribution which may be integrable, if $M = p - 1$ then the result is trivial. The next step is to solve the differential conditions (5). This is the difficult part and in general the best we can do is apply an EDS process which will identify the number and functional dependency of the solutions rather than explicitly construct solutions. We explain this process in the last section. In the event that we can construct a closed form solution we can, in the presence of a solvable structure, integrate the Frobenius integrable sub-distribution by using Theorem 8.

In the following section, we will consider the particular type of second order PDEs of one dependent variable and two independent variables and explain the algorithm in detail.

4. SECOND ORDER PDES

In this section, we examine two types of second order PDEs of one dependent variable and two independent variables. The first is of the form $u_{yy} = F(x, y, u, u_x, u_y, u_{xx}, u_{xy})$ and the second PDEs of evolution type.

4.1. PDEs of the type $u_{yy} = F(x, y, u, u_x, u_y, u_{xx}, u_{xy})$. Consider a second order PDE with one dependent real variable u and two independent real variables x, y given by

$$(6) \quad u_{yy} = F(x, y, u, u_x, u_y, u_{xx}, u_{xy}),$$

for some smooth function F . The embedded submanifold [20]

$$S := \{(x, y, u, u_x, u_y, u_{xx}, u_{xy}) \in J^2(\mathbb{R}^2, \mathbb{R}) \mid u_{yy} - F(x, y, u, u_x, u_y, u_{xx}, u_{xy}) = 0\},$$

is a subset of $J^2(\mathbb{R}^2, \mathbb{R})$. A local solution of the PDE is a 7-dimensional locus of $J^2(\mathbb{R}^2, \mathbb{R})$ described by the map $i : S \rightarrow J^2(\mathbb{R}^2, \mathbb{R})$, i.e.

$$i : (x, y, u, u_x, u_y, u_{xx}, u_{xy}) \mapsto (x, y, u, u_x, u_y, u_{xx}, u_{xy}, F).$$

We can study the solutions of (6) by studying the integral submanifolds N of the restricted contact system

$$(7) \quad \begin{aligned} D_V^\perp &:= \text{Sp}\{\theta^1 := du - u_x dx - u_y dy, \\ &\theta^2 := du_x - u_{xx} dx - u_{xy} dy, \\ &\theta^3 := du_y - u_{xy} dx - F dy\}, \end{aligned}$$

which project down to $X \subset \mathbb{R}^2$. If N satisfies the transverse condition $dx \wedge dy|_N \neq 0$ and has a tangent space that annihilates the restricted contact structure, then $i(N) \subset J^2(\mathbb{R}^2, \mathbb{R})$ is the 2-graph of a solution of (6). So, we need to study the vector field distribution dual of the pulled-back contact system, that is,

$$(8) \quad \begin{aligned} D_V &:= \text{Sp}\left\{V_1 := \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xy} \frac{\partial}{\partial u_y}, \right. \\ &V_2 := \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{xy} \frac{\partial}{\partial u_x} + F \frac{\partial}{\partial u_y}, \\ &\left. V_3 := \frac{\partial}{\partial u_{xx}}, V_4 := \frac{\partial}{\partial u_{xy}}\right\}. \end{aligned}$$

This dual distribution is called the Vessiot distribution D_V of (6).

4.1.1. Algorithm for finding the largest integrable sub-distributions of D_V .

In general the Frobenius integrability condition fails on the Vessiot distribution D_V . So, we will try to reduce the dimension of D_V and correspondingly add more 1-forms from $D^{\star\perp}$ to the co-distribution D_V^\perp as described at the end of the Section 3.

For this class of PDEs we need to add exactly two 1-forms to D_V^\perp because in adding a single 1-form ϕ the second algebraic condition $d\theta^2 \wedge \Omega_\theta \wedge \phi = 0$ implies that $\phi = 0$. Furthermore, we cannot add more than two 1-forms because the reduced Vessiot distribution cannot have dimension less than 2.

Step 1. Add two 1-forms $\phi^1, \phi^2 \in D^{\star\perp}$ in D_V^\perp .

We are interested in the largest integrable sub-distributions which satisfy the independence condition ($dx \wedge dy \neq 0$). So, for this purpose the coefficient of ψ^3 and ψ^4 in ϕ^1, ϕ^2 should be nonzero. Without a loss of generality we assume that

$$(9) \quad \phi^1 := \psi^3 - a_1 \psi^1 - a_2 \psi^2, \quad \phi^2 := \psi^4 - b_1 \psi^1 - b_2 \psi^2.$$

Now, solve the algebraic conditions

$$d\theta^a \wedge \Omega_\theta \wedge \Omega_\phi = 0, \quad a = 1 \dots 3.$$

Specifically,

$$a_2 = b_1, \quad b_2 = u_x F_u + u_{xx} F_{u_x} + a_1 F_{u_{xx}} + b_1 F_{u_{xy}} + u_{xy} F_{u_y} + F_x.$$

Step 2. Differential conditions:

Solve the differential conditions (5) i.e.

$$\begin{aligned} d\phi^1 \wedge \Omega_\theta \wedge \Omega_\phi &= 0, \\ d\phi^2 \wedge \Omega_\theta \wedge \Omega_\phi &= 0. \end{aligned}$$

Step 3. Integrate the reduced Vessiot distribution $D_{V_{red}}$ by using Theorem 8. To find the symmetries of reduced Vessiot distribution use DIMSYM, for example.

Remark 10. In applying the algorithm, there will exist situations when it may be difficult to solve the differential conditions (step 2). For such situations we will look on the alternative way to solve them. In Section 5, we will discuss the solution of these differential conditions in detail.

We illustrate this algorithm with an example:

Example 11. Consider the Laplace equation

$$u_{xx} + u_{yy} = 0$$

The Vessiot distribution and co-distribution is given by equations (7) and (8) with $F = -u_{xx}$. Now apply the algorithm step by step. By adding the two 1-forms (9) in the D_V^\perp and after solving the algebraic conditions (4) we obtain $a_2 = b_1$, $a_1 = -b_2$.

The corresponding differential constraints are

$$\begin{aligned} d\phi^1 \wedge \Omega_\theta \wedge \Omega_\phi &= u_x \frac{\partial b_1}{\partial u} - u_y \frac{\partial b_2}{\partial u} - \frac{\partial b_1}{\partial x} - \frac{\partial b_2}{\partial y} - u_{xx} \frac{\partial b_1}{\partial u_x} - u_{xy} \frac{\partial b_2}{\partial u_x} - u_{xy} \frac{\partial b_1}{\partial u_y} + u_{xx} \frac{\partial b_2}{\partial u_y} \\ &\quad + b_2 \frac{\partial b_1}{\partial u_{xx}} - b_1 \frac{\partial b_2}{\partial u_{xx}} - b_1 \frac{\partial b_1}{\partial u_{xy}} - b_2 \frac{\partial b_2}{\partial u_{xy}}, \\ d\phi^2 \wedge \Omega_\theta \wedge \Omega_\phi &= u_y \frac{\partial b_1}{\partial u} - u_x \frac{\partial b_2}{\partial u} - \frac{\partial b_2}{\partial x} + \frac{\partial b_1}{\partial y} + u_{xy} \frac{\partial b_1}{\partial u_x} - u_{xx} \frac{\partial b_2}{\partial u_x} - u_{xx} \frac{\partial b_1}{\partial u_y} - u_{xy} \frac{\partial b_2}{\partial u_y} \\ &\quad + b_1 \frac{\partial b_1}{\partial u_{xx}} + b_2 \frac{\partial b_2}{\partial u_{xx}} - b_2 \frac{\partial b_1}{\partial u_{xy}} - b_1 \frac{\partial b_2}{\partial u_{xy}}. \end{aligned}$$

In order to show some solutions we will choose an arbitrary value for these coordinates. For a particular choice of parameters we have the following solution.

$\phi^1 := \psi^3 - \psi^2, \phi^2 := \psi^4 - \psi^1$ satisfy the differential conditions and the transverse condition. The corresponding maximal dimension integrable sub-distribution D_{red} is generated by

$$\begin{aligned} \bar{V}_1 &:= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xy} \frac{\partial}{\partial u_y} + \frac{\partial}{\partial u_{xy}}, \\ \bar{V}_2 &:= \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{xy} \frac{\partial}{\partial u_x} - u_{xx} \frac{\partial}{\partial u_y} + \frac{\partial}{\partial u_{xx}}. \end{aligned}$$

For the integration of reduced Vessiot distribution use Theorem 8 and the following symmetries of D_{red}

$$\begin{aligned} X_1 &:= \frac{\partial}{\partial u}, & X_2 &:= -x \frac{\partial}{\partial u} - \frac{\partial}{\partial u_x}, & X_3 &:= -y \frac{\partial}{\partial u} - \frac{\partial}{\partial u_y}, \\ X_4 &:= -xy \frac{\partial}{\partial u} - y \frac{\partial}{\partial u_x} - x \frac{\partial}{\partial u_y} - \frac{\partial}{\partial u_{xy}}, & X_5 &:= -(y^2 - x^2) \frac{\partial}{\partial u} - 2y \frac{\partial}{\partial u_y} - 2x \frac{\partial}{\partial u_x} - 2 \frac{\partial}{\partial u_{xx}}, \end{aligned}$$

and after integration we obtain the following invariant functions

$$\begin{aligned} f^1 &:= u - xu_x + xyu_{xy} - yu_y + \frac{1}{2}(u_{xx}x^2 - u_{xx}y^2 - x^2y) + \frac{1}{6}y^3, \\ f^2 &:= u_x - yu_{xy} - xu_{xx} + xy, & f^3 &:= xu_{xy} - yu_{xx} - u_y + \frac{1}{2}(y^2 - x^2), \\ f^4 &:= u_{xy} - x, & f^5 &:= \frac{1}{2}(u_{xx} - y). \end{aligned}$$

The lifted solution on S is a common level set of these functions:

$$\{p \in S : f^\alpha(p) = c_\alpha\}.$$

This projects to $u = c_1 + c_2x - c_3y + c_4xy + \frac{1}{2}c_5(x^2 - y^2) + \frac{1}{2}x^2y - \frac{1}{6}y^3$.

Example 12. Consider the nonlinear second order partial differential equation (generalised equation of steady transonic gas flow) [14]

$$u_{yy} + A \frac{u_y}{y} + Bu_x u_{xx} = 0, \quad y \neq 0,$$

where A and B are arbitrary constants. For the sake of simplicity we choose $A = 1$ and $B = 1$. Now apply the algorithm step by step. By adding the two 1-forms (9) in the D_V^\perp and the solution of the algebraic conditions (4) implies $a_2 = b_1$, $b_2 = a_1u_x + u_{xx}^2 + \frac{u_{xy}}{y}$.

For a particular choice of parameters we find that

$$\phi^1 := \psi^3, \quad \phi^2 := \psi^4 + (u_{xx}^2 + \frac{u_{xy}}{y})dy$$

satisfy the differential conditions (we suppress domain considerations for clarity). The corresponding maximal dimension integrable sub-distribution D_{red} is generated by

$$\begin{aligned} \bar{V}_1 &:= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xy} \frac{\partial}{\partial u_y}, \\ \bar{V}_2 &:= \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{xy} \frac{\partial}{\partial u_x} + (u_x u_{xx} + \frac{u_y}{y}) \frac{\partial}{\partial u_y} - (u_{xx}^2 + \frac{u_{xy}}{y}) \frac{\partial}{\partial u_{xy}}. \end{aligned}$$

We now proceed to apply Theorem 8 to integrate the reduced Vessiot distribution. The symmetries of D_{red} are

$$\begin{aligned} X_1 &:= \frac{\partial}{\partial x}, & X_2 &:= 2x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 4u \frac{\partial}{\partial u} + 2u_x \frac{\partial}{\partial u_x} + 3u_y \frac{\partial}{\partial u_y} + u_{xy} \frac{\partial}{\partial u_{xy}}, & X_3 &:= \frac{\partial}{\partial u}, \\ X_4 &:= \log y \frac{\partial}{\partial u} + \frac{1}{y} \frac{\partial}{\partial u_y}, & X_5 &:= 2u \frac{\partial}{\partial u} - y \frac{\partial}{\partial y} + 2u_x \frac{\partial}{\partial u_x} + 3u_y \frac{\partial}{\partial u_y} + 3u_{xy} \frac{\partial}{\partial u_{xy}} + 2u_{xx} \frac{\partial}{\partial u_{xx}}. \end{aligned}$$

After integration, we obtain the following conserved quantities:

$$\begin{aligned}
f^1 &:= \frac{1}{4u_{xx}}(4xu_{xx} - 4u_x - y^2u_{xx}^2 + (y^2u_{xx}^2 + 2yu_{xy})(\log u_{xx} + 2\log y)), \\
f^2 &:= \frac{1}{2}(-\log y + \log u_{xx} - \log(2u_{xy} + yu_{xx}^2)), \\
f^3 &:= \frac{1}{64u_{xx}}(16yu_yu_{xx} - 16yu_xu_{xy} - 3y^4u_{xx}^4 - 8y^3u_{xx}^2u_{xy} + 4\log y), \\
f^4 &:= \frac{1}{64u_{xx}}(32u_x^2 + 7y^4u_{xx}^4 + 16y^3u_{xx}^2u_{xy} - (\log y)^2(8y^4u_{xx}^4 + 32y^3u_{xx}^2u_{xy} + 32y^2u_{xy}^2 - 16) \\
&\quad - \log y(64yu_xu_{xy} + 12y^4u_{xx}^4 + 32y^3u_{xx}^2u_{xy} - 16yu_yu_{xx})), \\
f^5 &:= \frac{1}{2}\log u_{xx}.
\end{aligned}$$

As before we can express the solution in the original coordinates.

4.2. Evolution equations. In this section we present an additive separable solution of the evolution equations. Consider the following second order evolution equation

$$(10) \quad u_x = F(x, y, u, u_y, u_{yy}),$$

with an extra condition $u_{xy} = 0$. This extra condition means the solution is additively separable. In [8] the authors say that a function has an additively separable form $u(x, y) = v(x) + w(y)$ if and only it satisfies the condition $u_{xy} = 0$.

The restricted contact distribution on \mathbb{R}^5 is

$$\begin{aligned}
D_V^\perp &:= \text{Sp}\{\theta^1 := du - Fdx - u_y dy, \\
&\quad \theta^2 := du_y - u_{yy} dy\}.
\end{aligned}$$

The corresponding dual distribution is

$$D_V := \text{Sp}\left\{V_1 := \frac{\partial}{\partial x} + F\frac{\partial}{\partial u}, V_2 := \frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + u_{yy}\frac{\partial}{\partial u_y}, V_3 := \frac{\partial}{\partial u_{yy}}\right\}.$$

In this case $D^{*\perp} := \text{Sp}\{\psi^1 = dx, \psi^2 = dy, \psi^3 = du_{yy}\}$.

Now follow the steps of the above algorithm:

If we add a 1-form $\phi \in D^{*\perp}$ to D_V^\perp and generate the algebraic conditions (4) for $\text{Sp}\{\theta^1, \theta^2, \phi\}$ to be Frobenius integrable we find a non zero solution. So, there is an integrable Vessiot sub-distribution of dimension 2. The solution is

$$\begin{aligned}
a_1 &= 0, \\
(11) \quad a_2 &= -\frac{F_y + u_y F_u + u_{yy} F_{u_y}}{F_{u_{yy}}}, \quad F_{u_{yy}} \neq 0.
\end{aligned}$$

For the evolution equations we have found the solution of algebraic conditions by adding one $\phi := \psi^3 - a_1\psi^1 - a_2\psi^2$ in the D_V^\perp . So the process is very simple in this case we need to go to the step 3 directly which is a differential condition. At this stage we need an expression for F . For this purpose we consider the potential Burgers equation.

Example 13. Consider the potential Burgers equation

$$u_x = u_{yy} + u_y^2,$$

with $u_{xy} = 0$. From (11), we have $a_1 = 0$, $a_2 = -2u_y$. This implies

$$\phi := \psi^3 + 2u_y\psi^2$$

which satisfy the differential condition and transverse condition.

The reduced Vessiot distribution is spanned by

$$D_{\text{red}} := \text{Sp}\{\bar{V}_1 := V_1, \bar{V}_2 := V_2 - 2u_y u_{yy} V_3\}.$$

The final step is the integration of reduced Vessiot distribution using Theorem 8.

The symmetries of D_{red} are

$$X_1 := \frac{\partial}{\partial y}, \quad X_2 := \frac{\partial}{\partial u}, \quad X_3 := 2x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - u_y \frac{\partial}{\partial u_y} - 2u_{yy} \frac{\partial}{\partial u_{yy}},$$

and after integration, we obtain

$$\begin{aligned} f^1 &:= y - \frac{1}{2}(u_y^2 + u_{yy})^{-\frac{1}{2}} \text{arctanh}((u_y^2 + u_{yy})^{-\frac{1}{2}} u_y), \\ f^2 &:= u - (u_y^2 + u_{yy})x + \frac{1}{2} \ln(u_{yy}) - \frac{1}{2} \ln(u_y^2 + u_{yy}), \\ f^3 &:= \frac{1}{2} \ln(u_y^2 + u_{yy}). \end{aligned}$$

The solution is $u = c_2 + x e^{2c_3} - \frac{1}{2} \ln(1 - \tanh^2(y e^{c_3} - c_1 e^{c_3}))$.

Remark 14. It is worth pointing out the utility of our geometric approach in the investigation of some natural questions concerning conserved quantities and symmetries. Having solved the problem of finding all the maximal integrable sub-distributions of the Vessiot distribution we can now ask, for example, whether there exists a solution of the PDE admitting a conserved quantity of energy type, or whether there exists a solution simultaneously admitting certain specified conserved quantities. We can also ask whether there exist solutions admitting a particular symmetry algebra, solutions which have certain topological properties and so on.

As an example, consider the Gaussian curvature of the graph of a solution of the PDE $u_{yy} = F(x, y, u, u_x, u_y, u_{xx}, u_{xy})$:

$$K = \frac{u_{xx}F - u_{xy}^2}{(1 + u_x^2 + u_y^2)^2}$$

and suppose that this solution is the projection of a common level set of five functions f^a of the type we have discussed earlier. Then K is clearly constant on the graph of the solution if and only if $dK \wedge df^1 \wedge \dots \wedge df^5 = 0$, expressing that K is a composite of f^1, \dots, f^5 . But less obvious and much more useful is that K is constant on the graph if and only if $dK \wedge \Omega_\theta \wedge \Omega_\phi = 0$. Moreover, if K is not constant on the graph then the one dimensional integral manifolds of $dK \wedge \Omega_\theta \wedge \Omega_\phi$ are the curves on the graph along which K is constant and their local existence is guaranteed.

Further calculus with K on the solution submanifold of $J^2(\mathbb{R}^2, \mathbb{R})$ is effected by using the tangent vector fields \bar{V}_1, \bar{V}_2 . Since \bar{V}_1, \bar{V}_2 project to the tangent directions to the graph of the solution, this calculus also projects. The point being that calculus on $J^2(\mathbb{R}^2, \mathbb{R})$ with functions like K is easier than on the base graph space.

5. SOLVING THE DIFFERENTIAL CONDITIONS

In this section we discuss the solution of differential conditions (5). We present two methods. The first is an Exterior Differential System (EDS) method and the second builds a solution invariant under a solvable structure.

5.1. EDS method.

From Proposition 2, we know that solving the Frobenius conditions

$$\begin{aligned} d\theta^a \wedge \Omega_\theta \wedge \Omega_\phi &= 0, & a = 1, \dots, 3 \\ d\phi^\alpha \wedge \Omega_\theta \wedge \Omega_\phi &= 0, & \alpha = 1, \dots, M \leq 2 \end{aligned}$$

for $\Omega_\phi := \phi^1 \wedge \dots \wedge \phi^M$ is equivalent to solving

$$(12) \quad d(\Omega_\theta \wedge \Omega_\phi) = \lambda \wedge \Omega_\theta \wedge \Omega_\phi$$

for Ω_ϕ and some 1-form λ . For later reference we note that, without loss, λ has no θ^a or ϕ^α components, and that $d\lambda \wedge \Omega_\theta \wedge \Omega_\phi = 0$. This last condition means that $d\lambda$ only has components with θ^a or ϕ^α as a factor.

Our approach here will be to create a differential ideal of 5-forms and then, through a closure condition, find those $\Omega_\theta \wedge \Omega_\phi$ which satisfy (12).

The standard EDS reference are the books [13, 15] and more details of EDS method by means of the inverse problem can be found in the work of [1]. We will give a brief summary of the method in this context.

The EDS process for finding the 5-forms satisfying (12) involves three steps. For the case of second order partial differential equations, we start with a module of 5-forms, Σ , and find the submodule of Σ^{final} that forms a differential ideal. We will find (or not) our Frobenius integrable 5-form in this ideal. The second step is to reformulate the Frobenius condition in terms of an equivalent linear Pfaffian system of one forms, and final step is to use the Cartan-Kähler theorem to determine the generality of the solutions of the problem.

The actual process for finding the submodule $\Sigma^{\text{final}} \subset \Sigma$ that generates the differential ideal is the following terminating recursive procedure. Starting with the submodule $\Sigma^0 := \Sigma$ find the submodule $\Sigma^1 \subseteq \Sigma^0$ such that for all $\Omega \in \Sigma^1$, $d\Omega \in \langle \Sigma^0 \rangle$, the ideal generated by Σ^0 .

We check if $\Sigma^1 = \Sigma^0$ and so is already a differential ideal. If not, we iterate the process, finding the submodule $\Sigma^2 \subset \Sigma^1 \subset \Sigma^0$ and so on until at some step, a differential ideal is found or the empty set is reached. If at any point during the process it is not possible to find a submodule containing simple forms, the problem has no solution.

Having found Σ^{final} , the next step in the EDS process is to express the problem of finding the Frobenius integrable 5-form in Σ^{final} as a Pfaffian system. Let the submodule Σ^{final} , be spanned by the set $\{\Omega^k : k = 1, \dots, d\}$, and calculate

$$d\Omega^k = \xi_h^k \wedge \Omega^h$$

to find the ξ_h^k 's which are now fixed one forms.

Let $\Omega = r_k \Omega^k$, now we are looking for those Ω 's in Σ^{final} such that $d\Omega = \lambda \wedge \Omega$, this gives:

$$\begin{aligned} d(r_k \Omega^k) &= \lambda \wedge r_k \Omega^k, \\ \Leftrightarrow dr_k \wedge \Omega^k + r_k d\Omega^k - r_k \lambda \wedge \Omega^k &= 0, \\ (13) \quad \Leftrightarrow (dr_k + r_h \xi_k^h - r_k \lambda) \wedge \Omega^k &= 0. \end{aligned}$$

So this leads us to the task of finding all the possible d -tuples of one forms $(\rho_k) = (\rho_1, \dots, \rho_d)$ such that

$$\rho_k \wedge \Omega^k = 0.$$

Suppose the solutions are

$$(\rho_k^A) := (\rho_1^A, \dots, \rho_d^A), \quad A = 1, \dots, e$$

with

$$\rho_k^A \wedge \Omega^k = 0,$$

that is, an e -dimensional module of d -tuples of 1-forms. Then equation (13) can be expressed as

$$\begin{aligned} (dr_k + r_h \xi_k^h - r_k \lambda) &= -p_A \rho_k^A, \\ (14) \quad \Leftrightarrow dr_k + r_h \xi_k^h - r_k \lambda + p_A \rho_k^A &= 0, \end{aligned}$$

where p_A are arbitrary functions.

At this stage, firstly the problem becomes that of solving (14) for r_k in terms of the unknown functions p_A , and secondly, identifying the restrictions on the choice of these p_A 's. The general method for finding the solution for this problem in EDS is to define an extended manifold $N := S \otimes \mathbb{R}^d \otimes \mathbb{R}^e$ with coordinates $\{x, y, u, u_x, u_y, u_{xx}, u_{xy}, r_k, p_A\}$, $k \in \{1, \dots, d\}$, $A \in \{1, \dots, e\}$ and look for 7-dimensional manifolds that are sections over S and on which the one forms

$$(15) \quad \sigma_k := dr_k + r_h \xi_k^h - r_k \lambda + p_A \rho_k^A$$

are zero. To find these manifolds, σ_k are considered constraint forms for a distribution on N whose integral submanifolds we want. We choose a basis of forms on N , $\{\alpha_a := dx, dy, du, du_x, du_y, du_{xx}, du_{xy}, \sigma_k, \pi_A\}$ where α_a are pulled-back basis for S , $\pi_A := dp_A$ and σ_k are defined above completes the basis. The condition that we want sections over S is that $\alpha_1 \wedge \alpha_2 \neq 0$. The Frobenius integrability condition for $D_\sigma = \text{Sp}\{\sigma_k\}$ is $d\sigma_k \equiv 0 \pmod{\sigma_k}$. But,

$$(16) \quad d\sigma_k \equiv \pi_k^i \wedge \alpha_i + t_k^{12} \alpha_1 \wedge \alpha_2 \pmod{\text{Sp}\{\sigma_k\}}$$

where the $\alpha_1 \wedge \alpha_2$ part is the *torsion* and π_k^i are some linear combination of dp_A . As $d\sigma_k$ expands with no $dp_A \wedge dp_B$ terms, the system is quasi-linear.

Because we want $\alpha_1 \wedge \alpha_2 \neq 0$ on the integral manifolds, we need to absorb all the *torsion* terms into the $\bar{\pi}_A = \pi_A - l_A^j \alpha_j$. If any of the *torsion* terms cannot be absorbed, then asking for $d\sigma_k \equiv 0 \pmod{\text{Sp}\{\sigma_k\}}$ is incompatible with the independence condition and therefore there is no solution.

Once the *torsion* terms have been removed, so that (16) becomes

$$(17) \quad d\sigma_k \equiv \bar{\pi}_k^i \wedge \alpha_i,$$

the next step is to create the tableau Π from (17), from which the Cartan characters can be calculated allowing us to apply the Cartan test for involution.

In order to satisfy the integrability condition we will force all the $\bar{\pi}_k^i$ terms to be zero. As a result we will obtain an over-determined system of linear algebraic equations for r_k .

We illustrate this idea with an example:

Example 15. Consider the second order Laplace equation $u_{yy} = -u_{xx}$. In order to produce an initial ideal we solve the algebraic equation (4) giving

$$a_2 = b_1, \quad a_1 = -b_2.$$

This means that Ω_ϕ is in the submodule, Σ^0 , of 2-forms generated by

$$\begin{aligned} \omega^1 &:= -\psi^1 \wedge \psi^2, \\ \omega^2 &:= \psi^1 \wedge \psi^4 + \psi^2 \wedge \psi^3, \\ \omega^3 &:= \psi^1 \wedge \psi^3 - \psi^2 \wedge \psi^4, \\ \omega^4 &:= \psi^3 \wedge \psi^4. \end{aligned}$$

Following the EDS process above, the first step is to find a differential ideal. This is done in an iterative process as follows.

Starting with $\Sigma^0 = \text{Sp}\{\Omega^k\}$, $k = 1, \dots, 4$, where $\Omega^k = \omega^k \wedge \Omega_\theta$, find the submodule Σ^1 where for each $\bar{\Omega} \in \Sigma^1$, $d\bar{\Omega}$ is in $\langle \Sigma^0 \rangle$, the ideal generated by Σ^0 . In this example we have $\Sigma^1 = \Sigma^0$. It means we already have a differential ideal and the process terminates.

So, now having a differential ideal of dimension four, we can continue with the EDS process. The next step is to solve $d(\omega^k \wedge \Omega_\theta) = \xi_l^k \wedge \omega^l \wedge \Omega_\theta$, for the one forms ξ_l^k . For this particular example we find that all $\xi_l^k = 0$.

The next step is to find 4-tuples of one forms (ρ_1, \dots, ρ_4) such that:

$$(18) \quad \rho_k \wedge \omega^k \wedge \Omega_\theta = 0,$$

Let $\rho_k = \hat{\rho}_k + \tilde{\rho}_k$, with $\hat{\rho}_k = \hat{a}_{ki}\psi^i$, $\tilde{\rho}_k = \tilde{a}_{ka}\theta^a$. In this case, the 16 \hat{a}_{ki} are real constants satisfying

$$\hat{a}_{23} = -\hat{a}_{34} - \hat{a}_{41}, \quad \hat{a}_{13} = -\hat{a}_{21} - \hat{a}_{32}, \quad \hat{a}_{31} = \hat{a}_{22} - \hat{a}_{14}, \quad \hat{a}_{42} = -\hat{a}_{33} + \hat{a}_{24},$$

and the 12 \tilde{a}_{ka} are arbitrary. So we have a 24 parameter family of solutions ρ_k^A , $A = 1, \dots, 24$, with $\hat{\rho}_k^A = 0$ for $A = 13, \dots, 24$ and $\tilde{\rho}_k^A = 0$ for $A = 1, \dots, 12$. Following the procedure from the outline, we now extend S to a new manifold with coordinates: $(x, y, u, u_x, u_y, u_{xx}, u_{xy}, r_k, p_A)$ and the problem can be written as that of finding the integrable distributions on N with $\sigma_k = 0$ where

$$(19) \quad \sigma_k := dr_k - r_k \lambda + p_A \rho_k^A,$$

along with independence condition $dx \wedge dy \neq 0$ on these distributions. Continuing the EDS process, set $\pi_A = dp_A$. Using this, a co-frame on N is $(dx, dy, du, du_x, du_y, du_{xx}, du_{xy}, \sigma_k, \pi_A)$. So the next step is to calculate $d\sigma_k$ modulo the ideal $\langle \sigma_k \rangle$ as follows:

Taking the exterior derivative of (19) and using this equation to remove resulting appearances of dr_k , we have

$$d\sigma_k \equiv (\pi_A - p_A \lambda) \wedge \rho_k^A - r_k d\lambda + p_A d\rho_k^A, \quad \text{mod } \{\sigma_k\}$$

As we know that we have a 24 parameter family of solutions, so we will consider the precise expressions for λ and $d\lambda$ for further simplification. Recall that λ has no θ^a components and

that $d\lambda$ has only $\theta \wedge \psi$ and $\psi \wedge \psi$ components. Let

$$\lambda := \lambda_i \psi^i \quad \text{and} \quad d\lambda_i := \lambda_{ij} \psi^j + \tilde{\lambda}_{ia} \theta^a.$$

$$\begin{aligned} d\sigma_1 \equiv & (\pi_1 - p_1\lambda - r_1 d\lambda_1) \wedge \psi^1 + (\pi_2 - p_2\lambda - r_2 d\lambda_2) \wedge \psi^2 + (\pi_4 + \pi_7 - (p_4 + p_7)\lambda - r_1 d\lambda_3) \wedge \psi^3 \\ & + (\pi_3 - p_3\lambda - r_1 d\lambda_4) \wedge \psi^4 + \sum_{A=13}^{24} [(\pi_A - p_A\lambda) \wedge \rho_1^A + p_A d\rho_1^A] \mod \{\sigma_k\}. \end{aligned}$$

In this case there exist r_k with $dr_k \in \text{Sp}\{\psi^i\}$ such that

$\lambda = (2 - u_{xx}\lambda_3 - u_{xy}\lambda_4)\psi^1 + (u_{xx}\lambda_4 - u_{xy}\lambda_3)\psi^2 + \lambda_3\psi^3 + \lambda_4\psi^4$, where λ_3, λ_4 are arbitrary, which can be seen by inspection from (19) with $p_A = 0, A = 13, \dots, 24$.

One possible solution:

$$r_1 = u_{xx}^2 + u_{xy}^2, \quad r_2 = -u_{xx}, \quad r_3 = u_{xy}, \quad r_4 = 1,$$

and

$$\Omega_\phi = -(u_{xx}^2 + u_{xy}^2)\psi^1 \wedge \psi^2 - u_{xx}(\psi^1 \wedge \psi^4 + \psi^2 \wedge \psi^3) + u_{xy}(\psi^1 \wedge \psi^3 - \psi^2 \wedge \psi^4) + \psi^3 \wedge \psi^4.$$

The distribution $\text{Sp}\{\theta^1, \theta^2, \theta^3, \phi^1, \phi^2\}$ is Frobenius integrable. We now proceed to apply the step 4 of the algorithm. The reduced Vessiot distribution is

$$\begin{aligned} \bar{V}_1 &:= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xy} \frac{\partial}{\partial u_y} + u_{xx} \frac{\partial}{\partial u_{xx}} + u_{xy} \frac{\partial}{\partial u_{xy}}, \\ \bar{V}_2 &:= \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{xy} \frac{\partial}{\partial u_x} - u_{xx} \frac{\partial}{\partial u_y} + u_{xy} \frac{\partial}{\partial u_{xx}} - u_{xx} \frac{\partial}{\partial u_{xy}}. \end{aligned}$$

After integrating the reduced Vessiot distribution, we have

$$\begin{aligned} f^1 &:= e^{-x}(u_{xx} \cos y - u_{xy} \sin y), & f^2 &:= e^{-x}(u_{xx} \sin y - u_{xy} \cos y), \\ f^3 &:= u - xu_x + xu_{xx} - u_{xx} + yu_{xy} - yu_y, & f^4 &:= u_y - u_{xy}, \quad f^5 := u_{xx} - u_x. \end{aligned}$$

This projects to

$$u = c_3 + c_4 y - c_5 x + e^x \frac{c_2 \sin y - c_1 \cos y}{\sin^2 y - \cos^2 y}.$$

5.2. The group invariance method. In this section we discuss the group invariance approach to solve the differential conditions on the ϕ^α :

$$(20) \quad d\phi^\alpha \wedge \Omega_\theta \wedge \Omega_\phi = 0$$

We apply steps 2 and 3 of the algorithm together.

Firstly we explain step 3 in this context. In applying Theorem 8, we need five symmetries in a solvable structure. Let $\bar{\Omega} = \Omega_\theta \wedge \Omega_\phi$ be a 5-form, where Ω_ϕ comes from the algebraic conditions. We wish $\bar{\Omega}$ to be Frobenius integrable and $\ker \bar{\Omega} = \text{Sp}\{\bar{V}_1, \bar{V}_2\}$ – the reduced Vessiot distribution. Now suppose there exists a solvable structure of 5 linearly independent vector fields X_1, \dots, X_5 . We will impose the condition $\bar{\Omega}(X_1, X_2, X_3, X_4, X_5) \neq 0$ and the conditions that X_1 is a symmetry of $\bar{\Omega}$, X_2 is a symmetry of $X_1 \lrcorner \bar{\Omega}$ and so on down to X_5 being a symmetry of $X_4 \lrcorner X_3 \lrcorner X_2 \lrcorner X_1 \lrcorner \bar{\Omega}$.

These symmetry conditions are

$$\begin{aligned}
 (21) \quad & \mathcal{L}_{X_1} \bar{\Omega} = \lambda_1 \bar{\Omega}, \\
 & \mathcal{L}_{X_2} (X_1 \lrcorner \bar{\Omega}) = \lambda_2 (X_1 \lrcorner \bar{\Omega}), \\
 & \vdots \\
 & \mathcal{L}_{X_5} (X_4 \lrcorner X_3 \lrcorner X_2 \lrcorner X_1 \lrcorner \bar{\Omega}) = \lambda_5 (X_4 \lrcorner X_3 \lrcorner X_2 \lrcorner X_1 \lrcorner \bar{\Omega}),
 \end{aligned}$$

for some smooth functions $\lambda_1, \dots, \lambda_5$. Note that at this stage we have not assumed the Frobenius integrability of $\bar{\Omega}$.

The way in which the symmetry conditions act as a catalyst to the solution of the differential conditions on Ω_ϕ can be seen more comprehensively by recalling that the sequence

$$(22) \quad \bar{\Omega}, X_1 \lrcorner \bar{\Omega}, X_2 \lrcorner X_1 \lrcorner \bar{\Omega}, \dots, X_4 \lrcorner \dots X_1 \lrcorner \bar{\Omega}$$

are all simple forms and that they are all Frobenius integrable if $\bar{\Omega}$ is Frobenius integrable. Hence the Frobenius integrability of each form in the sequence is a necessary condition for the Frobenius integrability of $\bar{\Omega}$. And since each form in the sequence has one-form factors in $Sp\{\theta^a, \phi^\alpha\}$ their Frobenius integrability represents a successive simplification of the Frobenius integrability of $\bar{\Omega}$.

We will now assume further that X_1, X_2, X_3 are linearly independent symmetries of Ω_θ satisfying $X_i \lrcorner \Omega_\theta \neq 0$, that is, symmetries of our PDE. And we insist that the first three symmetry conditions of (21) hold. For example, this means that

$$(23) \quad \mathcal{L}_{X_1} \Omega_\phi \equiv \mu_1 \Omega_\phi \pmod{\theta^a}$$

which imposes conditions on the ϕ^α . This is the essence of the technique.

The remaining two symmetries X_4, X_5 in (21) of necessity have non-zero components, X_4^V, X_5^V , in the kernel of Ω_θ . If $\ker \bar{\Omega} \oplus Sp\{X_1, X_2, X_3, X_4\}$ is Frobenius integrable then the one-form

$$(24) \quad \omega := \frac{X_4 \lrcorner (X_3 \lrcorner X_2 \lrcorner X_1 \lrcorner (\Omega_\theta \wedge \Omega_\phi))}{(\Omega_\theta \wedge \Omega_\phi)(X_1, \dots, X_5)},$$

is closed by virtue of (21) and it is a constraint form for $\bar{\Omega}$. This closure is a necessary condition for the integrability of $\ker \bar{\Omega}$. We can simplify this differential condition by writing ω as

$$\begin{aligned}
 \omega &= \frac{\Omega_\theta(X_1, X_2, X_3) X_4^V \lrcorner \Omega_\phi}{\Omega_\theta(X_1, X_2, X_3) \Omega_\phi(X_4^V, X_5^V)} \\
 &= \frac{X_4^V \lrcorner \Omega_\phi}{\Omega_\phi(X_4^V, X_5^V)} \in Sp\{\phi^1, \phi^2\}.
 \end{aligned}$$

(It should be noted that this *does not* imply that $\frac{X_5^V \lrcorner \Omega_\phi}{\Omega_\phi(X_4^V, X_5^V)}$ is closed mod ω).

In forcing the symmetry conditions (21) and the closure of ω we have found a closed form in Ω_ϕ and reduced the two differential conditions (20) to a single condition

$$(25) \quad d\phi \wedge \Omega_\theta \wedge \Omega_\phi = 0.$$

(Note that $\Omega_\phi = \phi^1 \wedge \phi^2$.) This result can clearly be generalised to any dimension but in general it isn't true that $Sp\{\phi^\alpha\}$ contains closed forms, even in this case. This is because

generically we don't have a solvable structure. The least we can expect is given in the following proposition

Proposition 16. Suppose that ϕ^α satisfies

$$\begin{aligned} d\theta^a \wedge \Omega_\theta \wedge \Omega_\phi &= 0, & a = 1, \dots, N, \\ d\phi^\alpha \wedge \Omega_\theta \wedge \Omega_\phi &= 0, & \alpha = 1, \dots, m. \end{aligned}$$

Then there exist $\tilde{\phi}^\alpha = l_\beta^\alpha \phi^\beta$, $|l_\beta^\alpha| \neq 0$ and h_a^α such that

$$(26) \quad df^\alpha = \tilde{\phi}^\alpha + h_a^\alpha \theta^a, \quad \alpha = 1, \dots, m.$$

Proof. Suppose that $\text{Sp}\{\theta^a, \phi^\alpha\}$ is Frobenius integrable,

$$\begin{aligned} d\theta^a \wedge \Omega_\theta \wedge \Omega_\phi &= 0, & a = 1, \dots, N, \\ d\phi^\alpha \wedge \Omega_\theta \wedge \Omega_\phi &= 0, & \alpha = 1, \dots, m. \end{aligned}$$

This implies that

$$df^a \in \text{Sp}\{\theta^a, \theta^\alpha\}, \quad \text{with} \quad df^1 \wedge \dots \wedge df^A \neq 0.$$

So, $df^A = h_a^A \theta^a + l_\alpha^A \phi^\alpha$, for some h_a^A, l_α^A . By relabelling the f^A and the ϕ^α we have

$$\begin{aligned} df^1 &= h_a^1 \theta^a + l_\alpha^1 \phi^\alpha, \\ &\vdots \\ df^m &= h_a^m \theta^a + l_\alpha^m \phi^\alpha, \quad \text{with} \quad |l_\beta^\alpha| \neq 0. \quad (\text{since } \phi^\alpha \in \text{Sp}\{df^A\}) \end{aligned}$$

Then put $\tilde{\phi}^\alpha = l_\beta^\alpha \phi^\beta$, $\alpha = 1, \dots, m$. So that $df^\alpha = h_a^\alpha \theta^a + \tilde{\phi}^\alpha$, $\alpha = 1, \dots, m$. □

We close this section with the following example.

Example 17. Consider the Gibbons-Tsarev equation [11]

$$(27) \quad u_{yy} = u_x u_{xy} - \beta u_y u_{xx} + \mu u + \nu.$$

We follow these authors by considering $\beta = 1$, $\mu = 0$ and $\nu = 1$. Now apply the step 1 of the algorithm. The solution of the algebraic equation (4) implies

$$a_2 = b_1, \quad b_2 = -a_1 u_y + b_1 u_x.$$

By substituting these values in the equation (9), we have

$$\phi^1 = \psi^3 - a_1 \psi^1 - b_1 \psi^2, \quad \phi^2 = \psi^4 - b_1 \psi^1 - (-a_1 u_y + b_1 u_x) \psi^2.$$

At this stage we have two unknown functions a_1 and b_1 . We impose the symmetry conditions (21) on $\bar{\Omega}$. The symmetries of the original PDE (27) are

$$\begin{aligned} X_1 &:= \frac{\partial}{\partial x}, & X_2 &:= \frac{\partial}{\partial u}, & X_3 &:= \frac{\partial}{\partial y}, \\ X_4 &:= -2y \frac{\partial}{\partial y} - 3x \frac{\partial}{\partial x} - 4u \frac{\partial}{\partial u} - 2u_y \frac{\partial}{\partial u_y} - u_x \frac{\partial}{\partial u_x} + 2u_{xx} \frac{\partial}{\partial u_{xx}} + u_{xy} \frac{\partial}{\partial u_{xy}}, \\ X_5 &:= x \frac{\partial}{\partial y} - 2x \frac{\partial}{\partial u} - 2 \frac{\partial}{\partial u_x} - u_x \frac{\partial}{\partial u_y} - u_{xx} \frac{\partial}{\partial u_{xy}}. \end{aligned}$$

The first three symmetry conditions (21) implies

$$\frac{\partial a_1}{\partial x} = 0, \quad \frac{\partial a_1}{\partial y} = 0, \quad \frac{\partial a_1}{\partial u} = 0, \quad \frac{\partial b_1}{\partial x} = 0, \quad \frac{\partial b_1}{\partial y} = 0, \quad \frac{\partial b_1}{\partial u} = 0$$

This means that a_1, b_1 are independent of x, y, u . The 4th and 5th symmetry conditions generate further 8 conditions. We have 8 first order partial differential equations for two unknown functions of 4 variables. The solution of this overdetermined system is

$$a_1 = 0, \quad b_1 = 0.$$

So, $\phi^1 = \psi^3, \phi^2 = \psi^4$ satisfy the differential conditions and the corresponding distribution is Frobenius integrable. For this case we do not need to impose the further necessary condition on Ω_ϕ .

The corresponding reduced Vessiot distribution is

$$\begin{aligned} \bar{V}_1 &:= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xy} \frac{\partial}{\partial u_y}, \\ \bar{V}_2 &:= \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{xy} \frac{\partial}{\partial u_x} + (u_x u_{xy} - u_y u_{xx} + 1) \frac{\partial}{\partial u_y}. \end{aligned}$$

After integration, we obtain

$$\begin{aligned} f^1 &:= x + y \frac{u_{xy}}{u_{xx}} - \frac{u_x}{u_{xx}}, \\ f^2 &:= -u + x u_x - x y u_{xy} - \frac{1}{2} x^2 u_{xx} + \frac{1}{2 u_{xx}^3} (2 y u_x u_{xy} u_{xx}^2 - y^2 u_{xy}^2 u_{xx}^2 - 2 y u_{xy}^2 u_{xx} \\ &\quad + 2 u_x u_{xy} u_{xx}^2 - 2 u_{xy}^2 - 2 u_y u_{xx}^2 + 2 u_{xx}), \\ f^3 &:= y + \frac{1}{u_{xx}} \log(u_x \frac{u_{xy}}{u_{xx}} - \frac{u_{xy}^2}{u_{xx}^2} + \frac{1}{u_{xx}} - u_y), \quad f^4 := -\frac{1}{2} \log u_{xx}, \quad f^5 := \frac{u_{xy}}{u_{xx}}. \end{aligned}$$

This projects to

$$u = -c_2 + x y c_4 c_5 - x c_1 c_4 - y c_1 c_4 c_5 - y c_5^2 + c_4^{-1} (y + e^{c_3 c_4 - y c_4}) + \frac{1}{2} c_4 (x^2 + y^2 c_5^2).$$

6. SUMMARY

In this paper, we have presented an algorithmic approach to the solution of second order partial differential equations of one dependent variable and two independent variables. We have successfully applied this algorithm to a wide class of linear and non-linear PDEs. Vessiot theory is used to generate the integrable sub-distributions which satisfy the appropriate independence condition. Then a solvable structure of each integrable sub-distribution is found (by using DIMSYM for example). For those solvable structures of sufficient dimension, we integrate the reduced Vessiot distribution giving a local solution of the original system as the parameterised integral submanifold of the sub-distribution. These solutions appear in the original coordinates of the problem. Different sub-distributions correspond to distinct solutions. In this process we subdivide the Frobenius condition into two parts, one being algebraic and one differential. Solving the former is straight forward, the most challenging task is to satisfy the differential conditions (step 2). For the situations where the differential conditions are not satisfied easily, we have given two approaches to solve the problem. One is

an application of the standard EDS method and other involves the simultaneous use of a solvable structure of symmetries to solve the conditions and to explicitly obtain the integrable submanifolds, giving a (generalised) group invariant solution. We give concrete examples throughout the paper.

Clearly future tasks involve the generalisation to higher order and higher dimensional systems, but we believe that the promise of symmetry reduction in the original coordinates and the calculus that is available for examining the shape of the solutions are exciting advances.

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DEPARTMENT OF MATHEMATICS AND STATISTICS, LA TROBE UNIVERSITY, VICTORIA, 3086, AUSTRALIA

E-mail address: ntehseen@students.latrobe.edu.au, g.prince@latrobe.edu.au